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Duffin and Schaeffer Type Inequality for Ultraspherical Polynomials*

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We show that

$$\|f^{(k)}\| \leqslant \left\| \frac{d^k}{dx^k} P_n^{(\alpha, \alpha)} \right\|, \qquad k=1, ..., n,$$

in the uniform norm for every real algebraic polynomial f of degree n which satisfies the inequalities

$$|f(x)| \leqslant |P_n^{(\alpha, \alpha)}(x)|$$

at the points x of local extrema of the ultraspherical polynomial $P_n^{(\alpha, \alpha)}$ in $[-1, 1]$. © 1996 Academic Press, Inc.

1. INTRODUCTION

Denote by π_n the set of all real algebraic polynomials of degree less than or equal to n. As usual, $T_n(x)$ denotes the Tchebycheff polynomial of the first kind, i.e.,

$$T_n(x) = \cos n \arccos x \qquad \text{for} \quad x \in [-1, 1].$$

Set

$$\|f\| := \max_{x \in [-1, 1]} |f(x)|.$$

In 1892 Vladimir Markov [2] proved the inequality

$$\|f^{(k)}\| \leqslant \|T_n^{(k)}\| \, \|f\|, \qquad k=1, ..., n,$$

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for every $f \in \pi_n$. A remarkable extension of this classical result was given by Duffin and Schaeffer [1]. They showed that

$$|f^{(k)}(x+iy)| \leqslant |T_n^{(k)}(1+iy)|, \qquad k=1, ..., n, \tag{1.1}$$

for every $x \in [-1, 1]$, $y \in (-\infty, \infty)$ and every polynomial $f \in \pi_n$ provided

$$|f(\eta_j^{(n)})| \leqslant 1, \qquad j=0, ..., n,$$

where $\eta_j^{(n)} := \cos(j\pi/n)$ are the points of local extrema of $T_n(x)$ in $[-1, 1]$.

Clearly (1.1) implies

$$\|f^{(k)}\| \leqslant \|T_n^{(k)}\|, \qquad k=1, ..., n. \tag{1.2}$$

Recently A. Shadrin [5] simplified the original proof of Markov and showed how inequality (1.1) can be deduced from Markov's work in the particular case $y=0$. Shadrin studied the more general question concerning the exact estimation of $\|f^{(k)}\|$ provided $|f(x)|$ is bounded by $|q(x)|$ at the set

$$-1 = t_0(q) < t_1(q) < \cdots < t_n(q) = 1$$

of extremal points of a given polynomial q from π_n $(q'(t_j)=0$, $j=1, ..., n-1)$. He proved the following

THEOREM A. *Let q be any fixed polynomial of degree n with n distinct zeros in $(-1, 1)$. Suppose that $f \in \pi_n$ and*

$$|f(t_j(q))| \leqslant |q(t_j(q))|, \qquad j=0, ..., n. \tag{1.3}$$

Then, for every $x \in [-1, 1]$ and $k=1, ..., n$,

$$|f^{(k)}(x)| \leqslant \max\left\{|q^{(k)}(x)|, \left|\frac{1}{k}(x^2-1)\, q^{(k+1)}(x) + xq^{(k)}(x)\right|\right\}. \tag{1.4}$$

Shadrin mentioned also that for $k=n$ and $k=n-1$ one can easily derive from (1.4) that the assumptions of Theorem A imply

$$\|f^{(k)}\| \leqslant \|q^{(k)}\|. \tag{1.5}$$

Does the inequality hold for every $k \in \{1, 2, ..., n\}$? Shadrin gave a simple counterexample which shows that (1.5) is not true in general for each admissible q and k. Despite of the efforts of many mathematicians no other example was found in which the conditions (1.3) imply (1.5) except the case $q=T_n$ given by Duffin and Schaeffer [1]. The purpose of this paper is to show that (1.5) holds if q is the ultraspherical polynomial $P_n^{(\alpha,\alpha)}$. Here we use the notation $P_n^{(\alpha,\beta)}$ from the book of Szegö [6] for the Jacobi polynomials. Precisely, $P_n^{(\alpha,\beta)}$ is the polynomial from π_n which is orthogonal in $[-1, 1]$ with weight

$$w(x) = (1-x)^\alpha (1+x)^\beta$$

to every polynomial of degree $n-1$ and normalized by the condition

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

Before concluding this section let us recall some of the basic properties of the Jacobi orthogonal polynomials which will be needed in the sequel.

Properties:

(i) $(d/dx)\, P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)\, P_{n-1}^{(\alpha+1,\beta+1)}(x)$;

(ii) for $\max\{\alpha, \beta\} \geqslant -1/2$ the supremum norm of $P_n^{(\alpha,\beta)}$ is attained at an endpoint of $[-1, 1]$; in the case $\alpha = \beta \geqslant -1/2$, $\|P_n^{(\alpha,\alpha)}\| = P_n^{(\alpha,\alpha)}(1) = \binom{n+\alpha}{n}$;

(iii) $y = P_n^{(\alpha,\beta)}$ satisfies the differential equation

$$(1-x^2)\, y'' + (\beta - \alpha - (\alpha+\beta+2)\, x)\, y' + n(n+\alpha+\beta+1)\, y = 0.$$

The proof of these facts can be found in Szegö [6] or any other textbook on orthogonal polynomials.

2. The Case $\alpha \geqslant -\frac{1}{2}$

We demonstrate here a very simple proof of Duffin and Schaeffer type inequality for a class of ultraspherical polynomials using Theorem A of A. Shadrin.

Theorem 2.1. *Let $t_j := t_j(P_n^{(\alpha,\alpha)})$, $j = 0, ..., n$, be the extremal points of $P_n^{(\alpha,\alpha)}$ in $[-1, 1]$ and $\alpha \geqslant -\frac{1}{2}$. Suppose that $f \in \pi_n$ and*

$$|f(t_j)| \leqslant |P_n^{(\alpha,\alpha)}(t_j)|, \qquad j = 0, ..., n.$$

Then

$$\|f^{(k)}\| \leqslant \left\| \frac{d^k}{dx^k} P_n^{(\alpha,\alpha)} \right\|$$

for all $k \in \{1, ..., n\}$.

Proof. It is well-known (see Rivlin [4], p. 158, Remark 1, or Szegö [6], pp. 95–96) that for $\alpha \geqslant -\frac{1}{2}$ the ultraspherical polynomials $P_n^{(\alpha,\alpha)}$ obey the representation

$$P_n^{(\alpha,\alpha)}(x) = \sum_{m=0}^{n} a_{n,m} T_m(x)$$

with nonnegative coefficients $\{a_{n,m}\}$. We shall use this fact to show that

$$\max_{x \in [-1,1]} \left| \frac{1}{k}(x^2-1)\, q^{(k+1)}(x) + xq^{(k)}(x) \right| = q^{(k)}(1) = \|q^{(k)}\| \tag{2.1}$$

if $q = P_n^{(\alpha,\alpha)}$. Indeed, it was shown in Shadrin's paper [5] that (2.1) holds for $q = T_m$, $m = 1, 2, \ldots$. Then for $q = P_n^{(\alpha,\alpha)}$ we have

$$\begin{aligned}
&\left| \frac{1}{k}(x^2-1)\, q^{(k+1)}(x) + xq^{(k)}(x) \right| \\
&\quad = \left| \frac{1}{k}(x^2-1) \sum_{m=0}^{n} a_{n,m} T_m^{(k+1)}(x) + x \sum_{m=0}^{n} a_{n,m} T_m^{(k)}(x) \right| \\
&\quad \leqslant \sum_{m=0}^{n} a_{n,m} \left| \frac{1}{k}(x^2-1)\, T_m^{(k+1)}(x) + xT_m^{(k)}(x) \right| \\
&\quad \leqslant \sum_{m=0}^{n} a_{n,m} \, |T_m^{(k)}(1)| = q^{(k)}(1).
\end{aligned}$$

Finally note that by Property (ii), $q^{(k)}(1) = \|q^{(k)}\|$ for $q = P_n^{(\alpha,\alpha)}$. It remains to apply Theorem A. The proof is completed. ∎

3. The Case $\alpha \geqslant -1$

Since the representation of $P_n^{(\alpha,\alpha)}$ in terms of Tchebycheff polynomials contains negative coefficients in the case $-1 < \alpha < -1/2$, we apply here another approach to extend the result of Theorem 2.1.

THEOREM 3.1. *Let $\alpha \geqslant -1$ and $f \in \pi_n$. Then the assumptions*

$$|f(t_j)| \leqslant |P_n^{(\alpha,\alpha)}(t_j)|, \qquad j = 0, \ldots, n,$$

imply

$$\|f^{(k)}\| \leqslant \left\| \frac{d^k}{dx^k} P_n^{(\alpha,\alpha)} \right\|.$$

for $k \in \{2, \ldots, n\}$.

As we mentioned already, the cases $k=n-1$ and $k=n$ follow immediately from Shadrin's result [5]. Thus we may stipulate in what follows that $1<k\leqslant n-2$.

The proof of Theorem 3.1 is based on several auxiliary propositions.

Denote $P_n^{(\alpha,\alpha)}$ by q, for simplicity, and set further

$$\begin{aligned} y(x) &:= q^{(k)}(x), \\ \varphi(x) &:= \frac{1}{k}(x^2-1)\, y'(x)+xy(x), \\ u(x) &:= \frac{1}{k}(x^2-1)\, y'(x). \end{aligned} \tag{3.1}$$

Let us point out here that according to Property (i)

$$y(x)=C.P_{n-k}^{(\alpha+k,\alpha+k)}(x)$$

with some positive constant C.

Observe that the function φ, we just defined, appears in the right hand side of the inequality (1.4) of Shadrin. Our goal is to show that

$$\|\varphi\|=\|y\|=y(1).$$

Then Theorem 3.1 could be derived easily from Shadrin's result. The next lemmas are steps towards this aim.

LEMMA 3.1. *The inequality*

$$\|\varphi\|\leqslant \max\{\|u\|, \|y\|\}$$

holds for all $k\in\{2, ..., n\}$, *if* $\alpha>-1$.

Proof. Clearly $|\varphi(\pm 1)|=y(1)=\|y\|$. Then in order to compare the norm of φ with that of y, it suffices to consider the values of $\varphi(x)$ at its critical points. If τ is such a point, i.e., if $\varphi'(\tau)=0$, then

$$(1-\tau^2)\, y''(\tau)-(k+2)\,\tau y'(\tau)-ky(\tau)=0.$$

On the other hand, since $y=C.P_{n-k}^{(\alpha+k,\alpha+k)}$, Property (iii) yields

$$(1-x^2)\, y''(x)-2(k+\alpha+1)\, xy'(x)+(n-k)(n+k+2\alpha+1)\, y(x)=0.$$

Combining this relation at $x=\tau$ with the previous one, we get

$$(k+2\alpha)\,\tau.y'(\tau)=[(n-k)(n+k+2\alpha+1)+k]\, y(\tau).$$

Thus the functions $x.y'(x)$ and $y(x)$ (and consequently $x.y(x)$ and $y'(x)$) have the same sign at the critical points of φ provided $k+2\alpha \geqslant 0$. For such k and α,

$$|\varphi(\tau)| \leqslant \max\left\{\frac{1}{k}(1-\tau^2)\,|y'(\tau)|, |\tau.y(\tau)|\right\}$$

and hence the proof is completed. ∎

The next lemma can be found in the book of Tricomi [7].

LEMMA 3.2 (Theorem of Sonin–Pòlya). *Let $u(x)$ be a nontrivial solution of the differential equation*

$$(pu')' + Pu = 0, \tag{3.2}$$

where the functions $p(x)$ and $P(x)$ are continuously differentiable in the interval $[a, b]$. Let $p(x)$ be positive on (a, b), $P(x)$ have no zeros on $[a, b]$ and let the function $p(x)\,P(x)$ be nondecreasing (nonincreasing) on $[a, b]$. Then the absolute values of the successive local extrema of u in (a, b) form a nonincreasing (nondecreasing) sequence.

We intend to apply Lemma 3.2 to the function u, defined by (3.1). In order to do this we shall need the following.

LEMMA 3.3. *The function $u(x) = (1/k)(x^2-1)\,y'(x)$ satisfies the differential equation*

$$\begin{aligned}((1-x^2)^{k+\alpha}\,u')' + (1-x^2)^{k+\alpha-2}\,[(n-k+1)\\ \times(n+k+2\alpha)(1-x^2) - 4(k+\alpha)]\,u = 0.\end{aligned} \tag{3.3}$$

The proof is a direct verification, using the fact that y and its derivatives are ultraspherical polynomials, and therefore satisfy the corresponding differential equations (see Property (iii)).

The next two conclusions from the previous lemmas describe the behavior of the function $u(x)$.

COROLLARY 3.1. *All critical points of the function $u(x) = (1/k)(x^2-1)\,y'(x)$ are located in the interval $(-\beta, \beta)$, where β is the positive root of the equation*

$$1-x^2 = \frac{4(k+\alpha)}{(n-k+1)(n+k+2\alpha)}.$$

Proof. Assume the contrary. Then there exists a point $\tau \notin (-\beta, \beta)$, such that $u'(\tau) = 0$. Assume that $\tau \in [\beta, 1)$. The case $\tau \in (-1, -\beta]$ is treated similarly. Denote by p and P the corresponding functions in the differential equation (3.3). Set

$$g(x) := p(x)\, u(x)\, u'(x).$$

Clearly, $g(1) = 0$, and since $g(\tau) = 0$, Rolle's theorem guarantees the existence of a point $\eta \in (\beta, 1)$, such that $g'(\eta) = 0$. But, using the differential equation (3.3), we get

$$\begin{aligned} g'(x) &= (p(x)\, u'(x))'\, u(x) + p(x)\, u'^2(x) = p(x)\, u'^2(x) - P(x)\, u^2(x) \\ &= (1-x^2)^{k+\alpha} \left\{ u'^2(x) - \frac{y'(x)^2}{k^2} [(n-k+1) \right. \\ &\quad \left. \times (n+k+2\alpha)(1-x^2) - 4(k+\alpha)] \right\} \end{aligned}$$

and therefore $g'(x) > 0$ for $x \in (\beta, 1)$, a contradiction. The corollary is proved. ∎

COROLLARY 3.2. *For $k \in \{2, ..., n-2\}$ and $\alpha > -1$, the local extrema of $|u|$ increase when $|x|$ increases.*

Proof. Because of the symmetry we study the function $|u(x)|$ only in the interval $[0, 1]$.

With p and P the corresponding functions in equation (3.3) we obtain

$$\begin{aligned} (pP)' = &-2x(1-x^2)^{2k+2\alpha-3} \{ [2(k+\alpha)-1](n-k+1)(n+k+2\alpha)(1-x^2) \\ &-8(k+\alpha)(k+\alpha-1) \}. \end{aligned}$$

It is seen that $(pP)'$ changes its sign at the point $x_0 \in (0, 1)$ satisfying the equality

$$1 - x_0^2 = \frac{8(k+\alpha)(k+\alpha-1)}{[2(k+\alpha)-1](n-k+1)(n+k+2\alpha)}.$$

Further, under the assumption of the proposition, we get

$$1 - x_0^2 < \frac{4(k+\alpha)}{(n-k+1)(n+k+2\alpha)},$$

which shows that $x_0 \in (\beta, 1)$. Therefore $(p(x)\, P(x))'$ does not change sign in the interval $[0, \beta)$, containing all non-negative critical points of u, and $(p(x)\,P(x))' < 0$ on this interval. The corollary then follows from Lemma 3.2. ▮

Denote by ξ the last critical point of u. According to Corollary 3.2

$$\|u\| = |u(\xi)|.$$

The next lemma gives some more information about the critical point ξ.

LEMMA 3.4. (a) $\xi > \sqrt{3}/3$;
(b) $y'(\xi) < (\sqrt{3}/3)\, y'(1)$.

Proof. (a) By definition, $u'(\xi) = 0$. This is equivalent to

$$(1-\xi^2)\, y''(\xi) = 2\xi y'(\xi). \tag{3.4}$$

Since ξ is located to the right of the last zero of y' and $y'(1) > 0$, we conclude that $y'(\xi) > 0$, $y'''(\xi) > 0$. Further, using the fact that $y' = C.P_{n-k-1}^{(k+\alpha+1,\, k+\alpha+1)}$ with $C > 0$, and the differential equation (iii), we get the relation

$$0 < (1-\xi^2)\, y'''(\xi) = 2(k+\alpha+2)\, \xi y''(\xi) - (n-k-1)(n+k+2\alpha+2)\, y'(\xi). \tag{3.5}$$

Now replacing $y''(\xi) = (2\xi/(1-\xi^2))\, y'(\xi)$ from (3.4) in the right hand side of (3. 5) and making use of the observation $y'(\xi) > 0$, we obtain

$$1-\xi^2 < \frac{4(k+\alpha+2)}{(n-k-1)(n+k+2\alpha+2)+4(k+\alpha+2)}. \tag{3.6}$$

But $k \leqslant n-2$ by assumption. This yields

$$1-\xi^2 < \frac{4(k+\alpha+2)}{(n+k+2\alpha+2)+4(k+\alpha+2)} \leqslant \frac{2}{3},$$

which leads to the assertion (a).

(b) Since y'' is an increasing function to the right of ξ,

$$y'(1) - y'(\xi) = y''(\theta)(1-\xi) > y''(\xi)(1-\xi) = \frac{2\xi}{1+\xi}\, y'(\xi)$$

(in the last equality we applied (3.4)). Hence

$$y'(\xi) < \frac{1+\xi}{1+3\xi}\, y'(1).$$

But $(1+x)/(1+3x)$ is a decreasing function of x in $(0, \infty)$. Using now the inequality $\xi > \sqrt{3}/3$ we get

$$y'(\xi) < \frac{1+\sqrt{3}/3}{1+\sqrt{3}}\, y'(1) = \frac{\sqrt{3}}{3}\, y'(1).$$

The lemma is proved. ∎

Note that Lemma 3.1 and Corollary 3.2 remain true also for $k=1$, if $\alpha \geqslant -1/2$.

Proof of Theorem 3.1. Let $\alpha > -1$. According to Theorem A, Lemma 3.1 and Corollary 3.2 the theorem will be proved if we show that $|u(\xi)| \leqslant y(1)$, i.e., if

$$\frac{1}{k}(1-\xi^2)\, y'(\xi) \leqslant y(1).$$

By Lemma 3.4, and particularly by (3.6), the last inequality will hold if

$$\frac{1}{k}\,\frac{4(k+\alpha+2)}{(n-k-1)(n+k+2\alpha+2)+4(k+\alpha+2)}\,\frac{\sqrt{3}}{3}\, y'(1) \leqslant y(1),$$

or, after dividing by a constant factor, if

$$\frac{2\sqrt{3}\,(k+\alpha+2)(n+k+2\alpha+1)}{3k[(n-k-1)(n+k+2\alpha+2)+4(k+\alpha+2)]}\, P_{n-k-1}^{(k+\alpha+1,\, k+\alpha+1)}(1)$$
$$\leqslant P_{n-k}^{(k+\alpha,\, k+\alpha)}(1).$$

In view of Property (ii) this is equivalent to

$$2\sqrt{3}\,(k+\alpha+2)(n-k)(n+k+2\alpha+1)$$
$$\leqslant 3k(k+\alpha+1)[(n-k-1)(n+k+2\alpha+2)+4(k+\alpha+2)].$$

Using the identity $(n-s)(n+s+2\alpha+1) = n(n+2\alpha+1) - s(s+2\alpha+1)$, after some straightforward calculations, the last inequality is reduced to

$$n(n+2\alpha+1) \geqslant (k+1)(k+2\alpha+2) - \frac{4(3k-\sqrt{3})(k+\alpha+1)(k+\alpha+2)}{3k(k+\alpha+1)-2\sqrt{3}\,(k+\alpha+2)},$$

which is obviously true for every $k \leqslant n-2$.

The case $\alpha = -1$ is obtained by going to the limit. The proof of the theorem is completed. ∎

Some computer experiments give us a reason to suggest that Theorem 3.1 is valid for $k=1$, too.

In the case $\alpha = -1$ the endpoints ± 1 are zeros of the ultraspherical polynomials and therefore the statement of Theorem 3.1 may be regarded as Duffin–Schaeffer's type inequality for polynomials satisfying zero boundary conditions. Precisely, the following theorem holds:

THEOREM 3.2. *Let P_{n-1} be the $(n-1)$-st Legendre polynomial with zeros $\{\xi_i\}_1^{n-1}$, $1 := \xi_0 > \xi_1 > \cdots > \xi_{n-1} > \xi_n := -1$. Let $Q_n(x) := (1-x^2)\, P'_{n-1}(x)$. If $P \in \pi_n$ satisfies the inequalities*

$$|p(\xi_i)| \leqslant |Q_n(\xi_i)|, \qquad i = 0, ..., n,$$

then

$$\|p^{(k)}\| \leqslant \|Q_n^{(k)}\| \qquad \textit{for} \quad k \in \{2, ..., n\}.$$

In order to verify this one only have to take into account that $P_n^{(-1,-1)}(x) = c(1-x^2)\, P'_{n-1}(x)$, $((1-x^2)\, P'_{n-1}(x))' = -n(n-1)\, P_{n-1}(x)$, and derive the claim as a corollary of Theorem 3.1.

Theorem 3.2 is close in spirit to a result of Rahman and Schmeisser ([3], Theorem 2), where T_{n-1} occurs instead of P_{n-1}.

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